Topic Of Nhochnhoc

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1 Problems

1. For positive reals a, b, c prove that:

$$(a+b+c)^3 \geq 6\sqrt{3}\,(a-b)(b-c)(c-a)$$

2. For $a, b, c \geq 0$ and $k \in \mathbf{R}$ find the best constant that satisfies

$$(a+b+c)^5 \geq k\left(a^2+b^2+c^2\right)(a-b)(b-c)(c-a)$$

3. For nonnegative reals a, b, c, find the best k satisfying

$$(a+b+c)^5 \geq k\,(ab+bc+ac)(a-b)(b-c)(c-a)$$

4. For nonnegative a, b, c, find the best k such that

$$\left(a^2+b^2+c^2\right)^3 \geq\ k\,(a-b)^2(b-c)^2(c-a)^2$$

5. For nonnegative reals a, b, c prove that

$$\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2} \leq \frac{1}{4}+\frac{4abc}{(a+b)(b+c)(c+a)}$$

6. For $a, b, c > 0$ satisfying $a^2+b^2+c^2=6$ Find P_{min} where

$$P=\frac{a}{bc}+\frac{2b}{ca}+\frac{5c}{ab}$$

7. For nonnegative reals a, b, c Prove that:

$$\frac{(a+b)^2(a+c)^2}{\left(b^2-c^2\right)^2}+\frac{(b+c)^2(a+b)^2}{\left(c^2-a^2\right)^2}+\frac{(b+c)^2(c+a)^2}{\left(a^2-b^2\right)^2} \geq 2$$

8. For positive reals a, b, c prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{16}{5}\cdot\frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{18}{5}$$

9. For positive a, b, c;show that :

$$\frac{\left(a^2+bc\right)\left(b^2+ca\right)\left(c^2+ab\right)}{\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)}+\frac{(a-b)(a-c)}{b^2+c^2}+\frac{(b-c)(b-a)}{c^2+a^2}+\frac{(c-a)(c-b)}{a^2+b^2} \geq 1$$

10. For positive reals a, b, c prove that :

$$1+\frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{16abc}{(a+b)(b+c)(c+a)}$$

11. For nonnegative a, b, c prove that :

$$\left(a^2+b^2+c^2-1\right)^2 \geq 2\left(a^3 b+b^3 c+c^3 a-1\right)$$

12. Let $a, b, c \geq 0$ satisfy $a+b+c=2$. Prove that we have;

- $\left(\sqrt{a^3}+\sqrt{b^3}\right)\left(\sqrt{b^3}+\sqrt{c^3}\right)\left(\sqrt{c^3}+\sqrt{a^3}\right) \leq 2;$
- $\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right) \leq 2;$
- $\left(\sqrt{a^5}+\sqrt{b^5}\right)\left(\sqrt{b^5}+\sqrt{c^5}\right)\left(\sqrt{c^5}+\sqrt{a^5}\right) \leq 2;$
- $\left(a^3+b^3\right)\left(b^3+c^3\right)\left(c^3+a^3\right) \leq 2;$
- $\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right) \leq (a+b)(b+c)(c+a).$

13. $a, b, c \geq 0$. Prove that

$$\left(a^2+5bc\right)\left(b^2+5ca\right)\left(c^2+5ab\right) \geq 27abc(a+b)(b+c)(c+a)$$

14. Let $a, b, c \geq 0$. Prove that:

$$\left(2a^2+7bc\right)\left(2b^2+7ca\right)\left(2c^2+7ab\right) \geq 27(ab+bc+ca)^3$$

15. $a, b, c \geq 0$ are the sides of a triangle. Prove that

$$a^3+b^3+c^3+9abc \leq 2[ab(a+b)+bc(b+c)+ca(c+a)]$$

16. Let $a, b, c>0$. Show that:

$$\frac{a^2}{2a^2+(b+c-a)^2}+\frac{b^2}{2b^2+(c+a-b)^2}+\frac{c^2}{2c^2+(a+b-c)^2} \leq 1$$

17. Let $a, b, c \geq 0$ Show that:

$$\frac{3a^2+5ab}{(b+c)^2}+\frac{3b^2+5bc}{(c+a)^2}+\frac{3c^2+5ca}{(a+b)^2} \geq 6$$

18. Let $a, b, c>0$ satisfy $a+b+c=3$. Prove that:

$$\left(a^3+b^3+c^3\right)(ab+bc+ca)^8 \leq 3^9$$

19. $a, b, c \geq 0$ Show that

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}+\frac{10abc}{(a+b)(b+c)(c+a)} \geq 2$$

20. For $a, b, c \geq 0$ such that $a+b+c=3$, prove the following inequality:

$$\left(ab^3+bc^3+ca^3\right)(ab+bc+ca) \leq 16$$

21. $a, b, c > 0$ satisfy $abc = 1$. Prove that

$$\frac{a}{\sqrt{b^2+2c}}+\frac{b}{\sqrt{c^2+2a}}+\frac{c}{\sqrt{a^2+2b}} \geq \sqrt{3}$$

22. For nonnegative a, b, c satisfying $ab+bc+ca=3$, prove that

$$3(a+b+c)+2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \geq 15$$

23. For nonnegative reals a, b, c prove that:

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}+\frac{1}{2} \geq \frac{5}{4} \cdot \frac{a^2+b^2+c^2}{ab+bc+ca}$$

24. For nonnegative a, b, c; show that

$$3\left(a^4+b^4+c^4\right)+7\left(a^2b^2+b^2c^2+c^2a^2\right) \geq 2\left(a^3b+b^3c+c^3a\right)+8\left(ab^3+bc^3+ca^3\right)$$

25. For $a, b, c \geq 0$, show that:

$$\frac{a^4}{(a+b)^4}+\frac{b^4}{(b+c)^4}+\frac{c^4}{(c+a)^4}+\frac{3abc}{2(a+b)(b+c)(c+a)} \geq \frac{3}{8}$$

26. Let $a, b, c \geq 0$ satisfy $a+b+c=3$ Prove that:

$$\sqrt[3]{\frac{a^3+4}{a^2+4}}+\sqrt[3]{\frac{b^3+4}{b^2+4}}+\sqrt[3]{\frac{c^3+4}{c^2+4}} \geq 3$$

27. For positive reals a, b, c show that:

$$5+\frac{3abc}{a^3+b^3+c^3} \geq 4\left(\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\right)$$

28. For nonnegative reals a, b, c prove that:

$$\frac{(a-b)^2}{(a+b)^2}+\frac{(b-c)^2}{(b+c)^2}+\frac{(c-a)^2}{(c+a)^2}+\frac{24(ab+bc+ca)}{(a+b+c)^2} \leq 8$$

29. Let $a, b, c \geq 0$; show that :

$$1+\frac{abc}{a^3+b^3+c^3} \geq \frac{32abc}{3(a+b)(b+c)(c+a)}$$

30. For nonnegative reals a, b, c show that:

$$\sqrt[3]{\frac{(a^2+bc)(b+c)}{a(b^2+c^2)}}+\sqrt[3]{\frac{(b^2+ca)(c+a)}{b(c^2+a^2)}}+\sqrt[3]{\frac{(c^2+ab)(a+b)}{c(a^2+b^2)}}\geq 3\sqrt[3]{2}$$

31. For nonnegative a, b, c show that

$$\frac{a}{\sqrt{b^2+bc+c^2}}+\frac{b}{\sqrt{c^2+ca+a^2}}+\frac{c}{\sqrt{a^2+ab+b^2}}\geq\frac{a+b+c}{\sqrt{ab+bc+ca}}$$

32. For nonnegative a, b, c show that

$$\sqrt[3]{\frac{a^5(b+c)}{(b^2+c^2)(a^2+bc)^2}}+\sqrt[3]{\frac{b^5(c+a)}{(c^2+a^2)(b^2+ca)^2}}+\sqrt[3]{\frac{c^5(a+b)}{(a^2+b^2)(c^2+ab)^2}}\geq\frac{3}{\sqrt[3]{4}}$$

33. If A, B, C are three angles of an acute triangle, find P_{min} where:

$$P=\frac{1}{\sin^n A}+\frac{1}{\sin^n B}+\frac{1}{\sin^n C}+\cos^m A\cos^m B\cos^m C$$

34. For nonnegative a, b, c show that

$$\left(a^2+5b^2\right)\left(b^2+5c^2\right)\left(c^2+5a^2\right)\geq 8abc\,(a+b+c)^3$$

35. For nonnegative reals a, b, c prove that:

$$1+\frac{8abc}{(a+b)(b+c)(c+a)}\geq\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

36. For nonnegative reals a, b, c prove that:

$$3+\frac{8abc}{(a+b)(b+c)(c+a)}\geq\frac{12(ab+bc+ca)}{(a+b+c)^2}$$

37. For nonnegative reals a, b, c prove that:

$$\sqrt{(a+b+c)(ab+bc+ca)}\geq\sqrt{abc}+\sqrt{\frac{(a+b)(b+c)(c+a)}{2}}$$

38. $a, b, c \geq 0$ Show that

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{1}{2}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

39. $a, b, c \geq 0$ Show that if $a+b+c=5$; we have:

$$10+ab^2+bc^2+ca^2\geq\frac{7}{8}\cdot\left(a^2b+b^2c+c^2a\right)$$

40. $a, b, c \geq 0$ Show that

$$\frac{a^2}{(a-b)^2} + \frac{b^2}{(b-c)^2} + \frac{c^2}{(c-a)^2} \geq\ 1$$

41. $a, b, c \geq 0$ Show that if they satisfy $a+b+c=3$ we always have:

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} + abc \geq\ 4$$

42. $a, b, c \geq 0$ Show that

$$\frac{5a^2+2bc}{(b+c)^2} + \frac{5b^2+2ca}{(c+a)^2} + \frac{5c^2+2ab}{(a+b)^2} \geq\ \frac{21}{4} \cdot \frac{a^2+b^2+c^2}{ab+bc+ca}$$

43. $a, b, c \geq 0$ Show that

$$\frac{3a^2+4bc}{(b+c)^2} + \frac{3b^2+4ca}{(c+a)^2} + \frac{3c^2+4ab}{(a+b)^2} \geq\ \frac{7}{4} \cdot \frac{(a+b+c)^2}{ab+bc+ca}$$

44. $a, b, c \geq 0$; $a+b+c = 2\sqrt[3]{12}$. Show that:

$$\sqrt[7]{1+a^3}\left(1+b^3\right)\left(1+c^3\right) \leq\ 169$$

45. For $a, b, c > 0$ satisfying $a+b+c=3$; show that:

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} + \frac{9abc}{4} \geq\ \frac{21}{4}$$

46. For $a, b, c > 0$ satisfying $a+b+c=3$, prove that if $k = \dfrac{10+4\sqrt{6}}{3}$ we have:

$$\sqrt{3\left(a^2+b^2+c^2\right)} + abc \geq\ 1+\sqrt{3k}$$

47. For $a, b, c > 0$ satisfying $a+b+c=3.$, show that

$$a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{c+a} \geq\ 3\sqrt{2}$$

48. For $a, b, c > 0$ satisfying $a+b+c=6$, Show that

$$\left(11+a^2\right)\left(11+b^2\right)\left(11+c^2\right) + 120abc \geq\ 4320$$

49. For $a, b, c > 0$ satisfying $ab+bc+ca=2$, show that

$$ab\left(4a^2+b^2\right) + bc\left(4b^2+c^2\right) + ca\left(4c^2+a^2\right) + 7abc\,(a+b+c) \geq\ 16$$

50. For $a, b, c > 0$, show that :

$$a\sqrt{a^2+3bc}+b\sqrt{b^2+3ca}+c\sqrt{c^2+3ab} \geq\ 2\left(ab+bc+ca\right)$$

51. For $a, b, c > 0$, prove that

$$a\sqrt{4a^2+5bc}+b\sqrt{4b^2+5ca}+c\sqrt{4c^2+5ab} \geq\ \left(a+b+c\right)^2$$

52. For positive real numbers a, b, c show that

$$\frac{a}{\sqrt{4a^2+5bc}}+\frac{b}{\sqrt{4b^2+5ca}}+\frac{c}{\sqrt{4c^2+5ab}} \leq\ 1$$

53. For positive real numbers a, b, c show that

$$\frac{a}{a+\sqrt{a^2+3bc}}+\frac{b}{b+\sqrt{b^2+3ca}}+\frac{c}{c+\sqrt{c^2+3ab}} \leq\ 1$$

54. For positive real numbers a, b, c such that $ab + bc + ca = 1$; show that

$$\frac{1}{\sqrt{a^2+b^2}}+\frac{1}{\sqrt{b^2+c^2}}+\frac{1}{\sqrt{c^2+a^2}} \geq\ 2+\frac{1}{\sqrt{2}}$$

55. For positive real numbers a, b, c satisfying $a + b + c = 2$; show that

$$\frac{1}{\sqrt{a^2+b^2}}+\frac{1}{\sqrt{b^2+c^2}}+\frac{1}{\sqrt{c^2+a^2}} \geq\ 2+\frac{1}{\sqrt{2}}$$

56. For positive real numbers a, b, c such that $ab + bc + ca = 3$, show that

$$\frac{a}{b^3+abc}+\frac{b}{c^3+abc}+\frac{c}{a^3+abc} \geq\ \frac{3}{2}$$

57. For positive real numbers a, b, c, show that

- $$\frac{a^2}{b^3+abc}+\frac{b^2}{c^3+abc}+\frac{c^2}{a^3+abc} \geq\ \frac{3}{2\left(a+b+c\right)}$$
- $$\sqrt{\frac{a^3}{b^3+abc}}+\sqrt{\frac{b^3}{c^3+abc}}+\sqrt{\frac{c^3}{a^3+abc}} \geq \frac{3}{\sqrt{2}}$$

58. For positive real numbers a, b, c that satisfy $a + b + c = 3$, show that

$$\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right) \geq\ \left(1+a\right)\left(1+b\right)\left(1+c\right)$$

59. For positive real numbers a, b, c , show that

$$\frac{a^3+b^3+c^3}{abc}+\frac{24abc}{\left(a+b\right)\left(b+c\right)\left(c+a\right)} \geq\ 4\cdot\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

60. For positive real numbers a, b, c , show that

$$a+b+c \geq \frac{a(b+c)}{\sqrt{a^2+3bc}}+\frac{b(c+a)}{\sqrt{b^2+3ca}}+\frac{c(a+b)}{\sqrt{c^2+3ab}}$$

61. For positive real numbers a, b, c , show that

$$a^k+b^k+c^k \geq \frac{a\left(b^k+c^k\right)}{\sqrt{a^2+3bc}}+\frac{b\left(c^k+a^k\right)}{\sqrt{b^2+3ca}}+\frac{c\left(a^k+b^k\right)}{\sqrt{c^2+3ab}}$$

62. For positive real numbers a, b, c , show that

$$\sqrt{a^2+4bc}+\sqrt{b^2+4ca}+\sqrt{c^2+4ab} \geq \sqrt{15(ab+bc+ca)}$$

63. For positive real numbers a, b, c , show that

$$\sqrt{\frac{a^3}{b^3+abc}}+\sqrt{\frac{b^3}{c^3+abc}}+\sqrt{\frac{c^3}{a^3+abc}} \geq \frac{3}{\sqrt{2}}$$

64. For positive real numbers a, b, c , show that

$$(a+b)^2(b+c)^2(c+a)^2 \geq \frac{64}{3}abc\left(a^2b+b^2c+c^2a\right)$$

65. For positive real numbers a, b, c , show that

$$\frac{3a^3+abc}{b^3+c^3}+\frac{3b^3+abc}{c^3+a^3}+\frac{3c^3+abc}{a^3+b^3} \geq 6$$

66. For positive real numbers a, b, c , show that

$$\left(a^2+b^2+c^2\right)(a+b+c) \geq 3\sqrt{3abc\left(a^3+b^3+c^3\right)}$$

67. For positive real numbers a, b, c , show that

$$\frac{1}{8a^2+bc}+\frac{1}{8b^2+ca}+\frac{1}{8c^2+ab} \geq \frac{1}{ab+bc+ca}$$

68. For positive reals a, b, c , show that

$$\frac{3\left(a^2+b^2+c^2\right)}{a+b+c} \geq \sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2} \geq \sqrt{3\left(a^2+b^2+c^2\right)}$$

69. For positive real numbers a, b, c , show that

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2} \leq 3$$

70. For positive real numbers a, b, c satisfying $a+b+c=3$, show that

$$\sqrt{a^2b+b^2c}+\sqrt{b^2c+c^2a}+\sqrt{c^2a+a^2b} \leq 3\sqrt{2}$$

71. For positive reals a, b, c , show that

$$\frac{3\left(a+b+c\right)}{2\left(ab+bc+ca\right)} \geq \frac{a}{a^2+b^2}+\frac{b}{b^2+c^2}+\frac{c}{c^2+a^2}$$

72. For positive reals a, b, c , show that

$$\frac{ab}{\sqrt{ab+2c^2}}+\frac{bc}{\sqrt{bc+2a^2}}+\frac{ca}{\sqrt{ca+2b^2}} \geq \sqrt{ab+bc+ca}$$

73. For positive reals a, b, c satisfying $a+b+c=3$, show that

$$\frac{a}{b^3+abc}+\frac{b}{c^3+abc}+\frac{c}{a^3+abc} \geq \frac{3}{2}$$

74. For positive reals a, b, c show that

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab} \leq \frac{3\sqrt{3}}{2\sqrt{abc\left(a+b+c\right)}}$$

75. Given that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9\left(a^3+b^3+c^3\right)}{\left(a^2+b^2+c^2\right)^2},$$

Prove that

$$\left(a^2+b^2+c^2\right)^2\left(ab+bc+ca\right) \geq 9abc\left(a^3+b^3+c^3\right)$$

76. For $a, b, c > 0$, show that

$$\frac{a^3+b^3+c^3}{abc}+\frac{24abc}{\left(a+b\right)\left(b+c\right)\left(c+a\right)} \geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

77. For $a, b, c > 0$, show that

$$\frac{\left(a+b+c\right)^2}{ab+bc+ca} \geq \frac{a\left(b+c\right)}{a^2+bc}+\frac{b\left(c+a\right)}{b^2+ca}+\frac{c\left(a+b\right)}{c^2+ab} \geq \frac{\left(a+b+c\right)^2}{a^2+b^2+c^2}$$

78. For $a, b, c > 0$, show that

$$\frac{\left(a+b+c\right)^2}{2\left(ab+bc+ca\right)} \geq \frac{a^2}{a^2+bc}+\frac{b^2}{b^2+ca}+\frac{c^2}{c^2+ab}$$

79. For $a, b, c > 0$, show that

$$3\left(a^2b + b^2c + c^2a\right)\left(ab^2 + bc^2 + ca^2\right) \geq abc\left(a + b + c\right)^3$$

80. For $a, b, c > 0$, show that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{\left(a + b + c\right)^3}{3\left(ab^2 + bc^2 + ca^2\right)}$$

81. For $a, b, c > 0$ satisfying $a + b + c = 3$, show that

- $\dfrac{a}{\sqrt{b + c^2}} + \dfrac{b}{\sqrt{c + a^2}} + \dfrac{c}{\sqrt{a + b^2}} \geq \dfrac{3}{\sqrt{2}}$
- $\dfrac{a}{b + c^2} + \dfrac{b}{c + a^2} + \dfrac{c}{a + b^2} \geq \dfrac{9}{3 + a + b + c}$

82. For $a, b, c > 0$, show that

$$\frac{1}{a\sqrt{a + b}} + \frac{1}{b\sqrt{b + c}} + \frac{1}{c\sqrt{c + a}} \geq \frac{3}{\sqrt{2abc}}$$

83. For $a, b, c > 0$, show that

$$\frac{ab}{\left(ab + 2c^2\right)^5} + \frac{bc}{\left(bc + 2a^2\right)^5} + \frac{ca}{\left(ca + 2b^2\right)^5} \geq \frac{1}{\left(ab + bc + ca\right)^4}$$

84. For positives a, b, c, prove that

- $\sqrt{\dfrac{a}{b + 3c}} + \sqrt{\dfrac{b}{c + 3a}} + \sqrt{\dfrac{c}{a + 3b}} \geq \dfrac{3}{2}$
- $\sqrt{\dfrac{a}{b + 2c}} + \sqrt{\dfrac{b}{c + 2a}} + \sqrt{\dfrac{c}{a + 2b}} \geq \sqrt[4]{8}$

85. For positives a, b, c, prove that

- $\dfrac{a}{\sqrt{ab + 8c^2}} + \dfrac{b}{\sqrt{bc + 8a^2}} + \dfrac{c}{\sqrt{ca + 8b^2}} \geq 1$
- $\dfrac{a}{\sqrt{ab + c^2}} + \dfrac{b}{\sqrt{bc + a^2}} + \dfrac{c}{\sqrt{ca + b^2}} \geq \dfrac{3}{\sqrt[3]{4}}$

86. Let $a, b, c \geq 1$, show that

$$\left(a + \frac{bc}{a^2}\right)\left(b + \frac{ca}{b^2}\right)\left(c + \frac{ab}{c^2}\right) \geq 27 \cdot \sqrt[3]{\left(a - 1\right)\left(b - 1\right)\left(c - 1\right)}$$

87. For positives a, b, c, prove that

$$\frac{a^{11}}{bc} + \frac{b^{11}}{ca} + \frac{c^{11}}{ab} + \frac{3}{a^2b^2c^2} \geq \frac{a^6 + b^6 + c^6 + 9}{2}$$

88. $x, y, z \in \left[0, \frac{1}{2}\right]$, show that:

$$\frac{x}{1+y^2} + \frac{y}{1+z^2} + \frac{z}{1+x^2} \le \frac{6}{5}$$

89. For positives a, b, c, prove that

$$a^3 + b^3 + c^3 + 3abc + 12 \ge 6(a+b+c)$$

90. For positives a, b, c satisfying $a + b + c = 2$, prove that

$$(1-ab)(1-bc)(1-ca) \ge \left(1-a^2\right)\left(1-b^2\right)\left(1-c^2\right)$$

91. For positives a, b, c, prove that

$$\left(1+\frac{4a}{a+b}\right)\left(1+\frac{4b}{b+c}\right)\left(1+\frac{4c}{c+a}\right) \le 27$$

92. For positives a, b, c, prove that

$$\left(a^2+b^2+c^2\right)(ab+bc+ca)^2 \ge \frac{27}{64}(a+b)^2(b+c)^2(c+a)^2$$

93. If A, B, C are the three angles of a triangle satisfying $5\cos A + 6\cos B + 7\cos C = 9$, show that we have:

$$\left(\sin\frac{A}{2}\right)^2 + \left(\sin\frac{B}{2}\right)^3 + \left(\sin\frac{C}{2}\right)^4 \ge \frac{7}{16}$$

94. Let $a, b, c \ge 0; ab + bc + ca + abc = 4$. Show that

$$\sqrt{a+3} + \sqrt{b+3} + \sqrt{c+3} \ge 6$$

95. For positives a, b, c, prove that

$$\frac{\sqrt{a^2+256bc}}{b+c} + \frac{\sqrt{b^2+256ca}}{c+a} + \frac{\sqrt{c^2+256ab}}{a+b} \ge 10$$

96. For positive real numbers a, b, c, prove that

$$\frac{a^5}{a+b} + \frac{b^5}{b+c} + \frac{c^5}{c+a} \ge \frac{a^3b^2}{a+b} + \frac{b^3c^2}{b+c} + \frac{c^3a^2}{c+a}$$

97. For positive real numbers a, b, c satisfying $ab + bc + ca = 1$, show that

$$a^3 + b^3 + c^3 + 3abc \ge 2abc(a+b+c)^2$$

98. For positive real numbers a, b, c satisfying $abc = 1$, show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 2\left(\frac{\sqrt{a}}{b+c} + \frac{\sqrt{b}}{c+a} + \frac{\sqrt{c}}{a+b}\right)$$

99. For positive real numbers a, b, c satisfying $ab + bc + ca = 1$, find the minimum value of:

$$\frac{ab\sqrt{ab}}{c} + \frac{bc\sqrt{bc}}{a} + \frac{ca\sqrt{ca}}{b}$$

100. For positive real numbers a, b, c satisfying $abc = 1$, prove that:

- $\frac{a}{a^2+3} + \frac{b}{b^2+3} + \frac{c}{c^2+3} \leq \frac{3}{4}$
- $\frac{a}{a^3+1} + \frac{b}{b^3+1} + \frac{c}{c^3+1} \leq \frac{3}{2}$
- $\frac{a}{2a^3+1} + \frac{b}{2b^3+1} + \frac{c}{2c^3+1} \leq 1$

Also, with the same conditions, prove or disprove that:

- $\frac{a}{(a+3)^2} + \frac{b}{(b+3)^2} + \frac{c}{(c+3)^2} \leq \frac{3}{16}$
- $\left(\frac{2a}{a^3+1}\right)^5 + \left(\frac{2b}{b^3+1}\right)^5 + \left(\frac{2c}{c^3+1}\right)^5 \leq 3$

101. For $a, b, c > 0$, prove the following inequality:

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{3}{abc} \geq \frac{12}{ab+bc+ca}$$

102. For $a, b, c \in [1, 2]$, prove that:

$$2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq \frac{a}{c} + \frac{c}{b} + \frac{b}{a} + 3$$

103. For $a, b, c \geq 0$, prove the following inequality:

$$\frac{a^2}{b^3+2abc} + \frac{b^2}{c^3+2abc} + \frac{c^2}{a^3+2abc} \geq \frac{3\sqrt{3(a^2+b^2+c^2)}}{(a+b+c)^2}$$

104. For $a, b, c > 0$ satisfying $ab + bc + ca = 2$, prove that

$$\sqrt{a^2+b^2+c^2} + 2 \geq a\sqrt{b^2+c^2} + b\sqrt{c^2+a^2} + c\sqrt{a^2+b^2}$$

105. For $a, b, c > 0$, prove that

$$\frac{a^2+b^2}{(a+b)^2} + \frac{b^2+c^2}{(b+c)^2} + \frac{c^2+a^2}{(c+a)^2} + \frac{a+b+c}{\sqrt{3(a^2+b^2+c^2)}} \geq \frac{5}{2}$$

106. For $a, b, c > 0$, prove that

$$\sqrt[9]{\frac{a^3}{b+c}} + \sqrt[9]{\frac{b^3}{c+a}} + \sqrt[9]{\frac{c^3}{a+b}} \geq \frac{1}{\sqrt[9]{2}} \left(\sqrt[9]{ab} + \sqrt[9]{bc} + \sqrt[9]{ca} \right)$$

107. For $a, b, c > 0$, $ab + bc + ca = 1$; find P_{min} where

$$P = \frac{1}{\sqrt{a^2 - ab + b^2}} + \frac{1}{\sqrt{b^2 - bc + c^2}} + \frac{1}{\sqrt{c^2 - ca + a^2}}$$

108. For $a, b, c > 0$, $ab + bc + ca \geq 11$; find P_{min} where

$$P = \sqrt{a^2 + 3} + \frac{\sqrt{7}}{5}\sqrt{b^2 + 3} + \frac{\sqrt{3}}{5}\sqrt{c^2 + 3}$$

109. Let a, b, c be positive reals satisfying $ab + bc + ca = 3$, Prove that:

$$\frac{1}{a^2b^3} + \frac{1}{b^2c^3} + \frac{1}{c^2a^3} + \frac{1}{3a^2 - 2ab + b^2} + \frac{1}{3b^2 - 2bc + c^2} + \frac{1}{3c^2 - 2ca + a^2} \geq \frac{9}{2}$$

110. For $a, b, c \geq 0$, prove that we always have:

- $\dfrac{a^2 + b^2 + c^2}{ab + bc + ca} + 15\sqrt{\dfrac{a^3b + b^3c + c^3a}{a^2b^2 + b^2c^2 + c^2a^2}} \geq \dfrac{47}{4};$
- $\sqrt{\dfrac{a^2 + b^2 + c^2}{ab + bc + ca}} + \sqrt{\dfrac{3\left(a^3b + b^3c + c^3a\right)}{a^2b^2 + b^2c^2 + c^2a^2}} \geq 1 + \sqrt{3}.$

111. $a, b, c > 0; a + b + c = 3$, Prove that:

$$\frac{1}{\sqrt{2a^2 + 1}} + \frac{1}{\sqrt{2b^2 + 1}} + \frac{1}{\sqrt{2c^2 + 1}} \geq \sqrt{3}$$

112. $a, b, c \geq 0; ab + bc + ca = 1$. Show that

$$\sqrt{a^2b + b^2c + c^2a} + \sqrt{ab^2 + bc^2 + ca^2} + 3\sqrt{abc} \geq 2$$

113. $a, b, c > 0; ab + bc + ca = 3$. Show that

$$\left(a + b^2\right)\left(b + c^2\right)\left(c + a^2\right) \geq 8$$

114. Prove that for all positive real numbers a, b, c we have:

- $\left(\dfrac{a}{b+c}\right)^3 + \left(\dfrac{b}{c+a}\right)^3 + \left(\dfrac{c}{a+b}\right)^3 + \dfrac{abc}{(a+b)(b+c)(c+a)} \geq \dfrac{1}{2} \cdot \left(\dfrac{a^2 + b^2 + c^2}{ab + bc + ca}\right)^2$
- $\left(\dfrac{a}{b+c}\right)^3 + \left(\dfrac{b}{c+a}\right)^3 + \left(\dfrac{c}{a+b}\right)^3 + \dfrac{5abc}{(a+b)(b+c)(c+a)} \geq \dfrac{a^2 + b^2 + c^2}{ab + bc + ca}$

115. $a, b, c \geq 0$; prove that

$$\frac{2\left(a^2+b^2+c^2\right)}{(ab+bc+ca)^2} \geq \frac{a^2+b^2}{\left(a^2+ab+b^2\right)^2}+\frac{b^2+c^2}{\left(b^2+bc+c^2\right)^2}+\frac{c^2+a^2}{\left(c^2+ca+a^2\right)^2}$$

116. $a, b, c \geq 0$; prove that

$$\frac{a+b}{\sqrt{a^2+ab+b^2}}+\frac{b+c}{\sqrt{b^2+bc+c^2}}+\frac{c+a}{\sqrt{c^2+ca+a^2}} \geq 2\sqrt{3} \cdot \left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^{\frac{4}{9}}$$

117. $a, b, c \geq 0$; prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{9}{4} \cdot \frac{abc}{a^3+b^3+c^3} \geq \frac{15}{4}$$

118. For $x, y, z \geq 0$, prove that we always have :

$$32xyz(x+y+z)\left(x^2+y^2+z^2+xy+xz+yz\right) \leq 9(x+y)^2(x+z)^2(y+z)^2$$

119. $a, b, c \geq 0$. Prove that:

$$\frac{11}{3}(a+b+c) \geq 8\sqrt[3]{abc}+3\sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

120. $a, b, c > 0$ satisfy $a+b+c=3$, prove that:

$$\frac{a}{\sqrt{4b+4c^2+1}}+\frac{b}{\sqrt{4c+4a^2+1}}+\frac{c}{\sqrt{4a+4b^2+1}} \geq 1$$

121. $a+b+c=3; a, b, c > 0$. Show that:

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\frac{6}{ab+bc+ca} \geq 5$$

122. $ab+bc+ca=3; a, b, c > 0$. Prove that

$$\sqrt{a^2+a}+\sqrt{b^2+b}+\sqrt{c^2+c} \geq 3\sqrt{2}$$

123. $a, b, c > 0; abc=1$, show that, for $k=10, 11$ we have

$$a+b+c \geq 3\sqrt[k]{\frac{a^3+b^3+c^3}{3}};$$

124. Let $a, b, c > 0$ satisfy $a+b+c=3$. Prove that

$$\sqrt[3]{a^2+ab+bc}+\sqrt[3]{b^2+bc+ca}+\sqrt[3]{c^2+ca+ab} \geq \sqrt[3]{3}(ab+bc+ca)$$

125. Given $a, b, c > 0$, Prove that:

$$\frac{1}{(3a+2b+c)^2}+\frac{1}{(3b+2c+a)^2}+\frac{1}{(3c+2a+b)^2}\le\frac{9}{4(ab+bc+ca)}$$

126. Let $a, b, c > 0$. Prove that:

$$a^2+b^2+c^2+\frac{\sqrt{3}(ab+ac+bc)\sqrt[3]{abc}}{\sqrt{a^2+b^2+c^2}}\ge 2(ab+bc+ca)$$

127. Given $a, b, c > 0$ and $abc = 1$, Prove that

$$81\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right)\le 8(a+b+c)^4$$

128. Given $a, b, c \ge 0$ and $a + b + c = 2$, Prove that:

$$\frac{\sqrt[4]{a^2+6ab+b^2}}{a+b}+\frac{\sqrt[4]{b^2+6bc+c^2}}{b+c}+\frac{\sqrt[4]{c^2+6ca+a^2}}{c+a}\ge 2+\frac{1}{\sqrt[4]{2}}$$

129. Given $a, b, c > 0$, Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{2a+b}+\frac{3}{2b+c}+\frac{3}{2c+a}$$
$$\ge 2\sqrt{3}\left(\frac{1}{\sqrt{a(a+2b)}}+\frac{1}{\sqrt{b(b+2c)}}+\frac{1}{\sqrt{c(c+2a)}}\right)$$

130. For $a, b, c > 0$, we have:

$$\frac{(a+b)^2}{a+b+2c}+\frac{(b+c)^2}{b+c+2a}+\frac{(c+a)^2}{c+a+2b}\ge\sqrt{3\left(a^2+b^2+c^2\right)}$$

131. For $a, b, c > 0; ab + bc + ca = 3$, prove that:

$$\frac{a^2+bc+4ab}{a+8b}+\frac{b^2+ca+4bc}{b+8c}+\frac{c^2+ab+4ca}{c+8a}\ge 2$$

132. Given a, b, c are three real numbers satisfying $a + b + c = 3$, Prove that:

$$\frac{a^2+b^2c^2}{(b-c)^2}+\frac{b^2+c^2a^2}{(c-a)^2}+\frac{c^2+a^2b^2}{(a-b)^2}\ge 5$$

133. Given $a, b, c > 0$, prove that:

$$\sqrt[4]{\frac{a^3+b^3+c^3}{abc}}\ge 2\sqrt[4]{3}\left(\frac{a}{a+2b+3c}+\frac{b}{b+2c+3a}+\frac{c}{c+2a+3b}\right)$$

134. Given $a, b, c \geq 0$, show that:

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{8(ab+bc+ca)}{(a+b+c)^4} \geq \frac{11}{4(ab+bc+ca)}$$

135. Given $a, b, c \geq 0$, prove that:

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq 3 \cdot \sqrt[3]{\frac{2(a^3+b^3+c^3)+abc}{7}}$$

136. Let $a, b, c > 0$. Prove that the following holds good for $k = \frac{8}{3}$. Also, find the best constant k such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + k \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \geq k + 3.$$

137. Let $a, b, c > 0$. Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{6\left(a^2+b^2+c^2\right) - 3(ab+bc+ca)}{a+b+c}$$

138. Given $a, b, c > 0$, show that:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3 \cdot \sqrt{\frac{a^4+b^4+c^4}{a^2+b^2+c^2}}$$

139. $a, b, c \geq 0$; prove that

$$\frac{a}{\sqrt{b^2+ab+9c^2}} + \frac{b}{\sqrt{c^2+bc+9a^2}} + \frac{c}{\sqrt{a^2+ca+9b^2}} \geq \frac{3}{\sqrt{11}}$$

179, For $a, b, c > 0$ such that $a+b+c=3$, prove the following inequality:

- $$\frac{(1+a)^2(1+b)^2}{1+c^2} + \frac{(1+b)^2(1+c)^2}{1+a^2} + \frac{(1+c)^2(1+a)^2}{1+b^2} \geq 24$$
- $$\frac{(b+c)^5+32}{a^3+1} + \frac{(c+a)^5+32}{b^3+1} + \frac{(a+b)^5+32}{c^3+1} \geq 96$$

140. For $a, b, c > 0$, prove the following inequality:

$$\frac{(a+b)(b+c)(c+a)}{8abc} \geq \frac{\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)}{(a^2+bc)(b^2+ca)(c^2+ab)}$$

141. For $a, b, c > 0$ such that $ab+bc+ca=3$, prove the following inequality:

$$(a+2b)(b+2c)(c+2a) \geq 8$$

142. For $a, b, c > 0$, prove the following inequality:

$$21 + \frac{a^3 + b^3 + c^3}{abc} \geq \frac{8(a+b+c)}{\sqrt[3]{abc}}$$

143. For $a, b, c > 0$, prove the following inequality:

$$33 + \frac{(a+b+c)\left(a^2+b^2+c^2\right)}{abc} \geq \frac{14(a+b+c)}{\sqrt[3]{abc}}$$

144. For $a, b, c > 0$ and $k \geq 3$, prove the following inequality:

$$12k - 9 + \frac{(a+b+c)\left(a^2+b^2+c^2\right)}{abc} \geq \frac{4k\left(\sqrt[k]{a^3} + \sqrt[k]{b^3} + \sqrt[k]{c^3}\right)}{\sqrt[k]{abc}}$$

145. For $a, b, c > 0$, prove the following inequality:

$$\frac{\sqrt{a^2+3bc}}{(b+c)(a+8b)} + \frac{\sqrt{b^2+3ca}}{(c+a)(b+8c)} + \frac{\sqrt{c^2+3ab}}{(a+b)(c+8a)} \geq \frac{1}{a+b+c}$$

146. For $a, b, c > 0$ satisfying $ab + bc + ca = 3$, prove the following:

- $\sqrt{a^2+a} + \sqrt{b^2+b} + \sqrt{c^2+c} \geq 3\sqrt{2}$
- $\sqrt[3]{5a^3+3a} + \sqrt[3]{5b^3+3b} + \sqrt[3]{5c^3+3c} \geq 6$

147. For $a, b, c > 0$, prove the following inequality:

$$\frac{\sqrt{ab}}{ab+c^2} + \frac{\sqrt{bc}}{bc+a^2} + \frac{\sqrt{ca}}{ca+b^2} \geq \frac{9\left(a^3+b^3+c^3\right)}{2(a+b+c)^4}$$

148. Given $a, b, c > 0$, Prove that:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \sqrt[4]{3} \cdot \sqrt[4]{\frac{a^3+b^3+c^3}{abc}}.\sqrt{a^2+b^2+c^2}$$

149. Given $a, b, c > 0$, Prove that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.\sqrt[4]{\frac{a^3+b^3+c^3}{3abc}}$$

150. Given $a, b, c \geq 0$. Prove that:

$$\sqrt[3]{a^2+4bc} + \sqrt[3]{b^2+4ca} + \sqrt[3]{c^2+4ab} \geq \sqrt[3]{45(ab+bc+ca)}$$

151. For positives a, b, c, prove that

$$\frac{16abc-3a^3}{(b-c)^2} + \frac{16abc-3b^3}{(c-a)^2} + \frac{16abc-3c^3}{(a-b)^2} \geq 21(a+b+c)$$

152. For positives a, b, c, prove that

$$\frac{abc}{(a+b)(b+c)(c+a)} \leq \frac{(a+b)(a+b+2c)}{(3a+3b+2c)^2} \leq \frac{1}{8}$$

153. In $\triangle ABC$ show that

$$\left(\frac{\cos\frac{A}{2}}{\tan A}\right)^2 + \left(\frac{\cos\frac{B}{2}}{\tan B}\right)^2 + \left(\frac{\cos\frac{C}{2}}{\tan C}\right)^2 + \frac{3}{4}.\frac{\sin A+\sin B+\sin C}{\tan A+\tan B+\tan C} \geq \frac{9}{8}$$

154. $a, b, c \geq 0; a + 2b + 3c = 4$. Prove that:

$$\left(a^2b + b^2c + c^2a + abc\right)\left(ab^2 + bc^2 + ca^2 + abc\right) \leq 8$$

155. For positives a, b, c such that $ab + bc + ca \geq 11$; prove that

$$\sqrt[3]{a^2+3} + \frac{7}{5\sqrt[3]{14}} \cdot \sqrt[3]{b^2+3} + \frac{\sqrt[3]{9}}{5} \cdot \sqrt[3]{c^2+3} \geq \frac{23}{5\sqrt[3]{2}}$$

156. Let a, b, c be three real numbers satisfying $a + b + c = 3$. Prove that

$$\frac{a+b}{3a^2+b^2+c^2+3} + \frac{b+c}{3b^2+c^2+a^2+3} + \frac{c+a}{3c^2+a^2+b^2+3} \leq \frac{3}{4}$$

157. Let a, b, c be three real numbers satisfying$a + b + c = 3$. Prove that

$$\left(\frac{a^2-bc}{b-c}\right)^2 + \left(\frac{b^2-ca}{c-a}\right)^2 + \left(\frac{c^2-ab}{a-b}\right)^2 \geq 18$$

158. Let a, b, c be real numbers. Prove that

$$\frac{a^4}{(b-c)^2} + \frac{b^4}{(c-a)^2} + \frac{c^4}{(a-b)^2} \geq 2(ab+bc+ca)$$

159. Let a, b, c be three real numbers. Prove that

$$\left|\frac{a^2+bc}{b-c}\right| + \left|\frac{b^2+ca}{c-a}\right| + \left|\frac{c^2+ab}{a-b}\right| \geq \sqrt{3(a^2+b^2+c^2)}$$

160. Given $a, b, c \geq 0$, Find Minimum of the following expression.

$$P = \frac{a}{b+c} + m \cdot \sqrt{\frac{b}{c+a}} + n \cdot \sqrt[3]{\frac{c}{a+b}};$$

Where $m = \dfrac{9}{8}$ and $n = \dfrac{3}{2\sqrt[3]{2}}$.

161. Let a, b, c be positive real numbers. Prove that:

$$\frac{2a+b}{a+2b}\sqrt{\frac{c}{2b+c}}+\frac{2b+c}{b+2c}\sqrt{\frac{a}{2c+a}}+\frac{2c+a}{c+2a}\sqrt{\frac{b}{2a+b}}\geq\sqrt{3}$$

162. Let a, b, c be positive real numbers and $n \in \mathbb{N}$. Prove that

$$\sqrt[n]{\frac{a}{a+(n-1)\,b}}+\sqrt[n]{\frac{b}{b+(n-1)\,c}}+\sqrt[n]{\frac{c}{c+(n-1)\,a}}\leq\frac{3}{\sqrt[n]{n}}$$

163. Let a, b, c be three nonnegative real numbers satisfying$a+b+c=1$ Prove that:

$$\left(a^2+ab+bc\right)\left(b^2+bc+ca\right)\left(c^2+ca+ab\right)-2abc\leq\frac{108}{3125}$$

164. Let a, b, c be positive real numbers. Prove that:

$$\left(\frac{a}{4a+5b+3c}\right)^{\frac{2}{3}}+\left(\frac{b}{4b+5c+3a}\right)^{\frac{2}{3}}+\left(\frac{c}{4c+5a+3b}\right)^{\frac{2}{3}}\leq\frac{3}{12^{\frac{2}{3}}}$$

165. Let a, b, c be three positive real numbers. Prove that:

$$\sqrt{\frac{a}{a+2b+3c}}+\sqrt{\frac{b}{b+2c+3a}}+\sqrt{\frac{c}{c+2a+3b}}\leq\sqrt{\frac{3}{2}}$$

166. Let a, b, c be positive real numbers. Prove that:

$$\frac{a\,(b+c)}{b^2+c^2}+\frac{b\,(c+a)}{c^2+a^2}+\frac{c\,(a+b)}{a^2+b^2}+3\geq4\left(\frac{ab}{ab+c^2}+\frac{bc}{bc+a^2}+\frac{ca}{ca+b^2}\right)$$

167. Let a, b, c be positive real numbers. Prove that:

$$\frac{1}{a+\sqrt{a^2+3b^2}}+\frac{1}{b+\sqrt{b^2+3c^2}}+\frac{1}{c+\sqrt{c^2+3a^2}}\geq\frac{2}{a+b+c}+\frac{a+b+c}{3\left(a^2+b^2+c^2\right)}$$

168. Let a, b, c be positive real numbers. Prove that:

$$\left(\frac{a}{13a+17b}\right)^{\frac{2}{5}}+\left(\frac{b}{13b+17c}\right)^{\frac{2}{5}}+\left(\frac{c}{13c+17a}\right)^{\frac{2}{5}}\leq\frac{3}{30^{\frac{2}{5}}}$$

169. Let a, b, c be nonnegative real numbers. Prove that:

$$\frac{1}{2a^2+bc}+\frac{1}{2a^2+bc}+\frac{1}{2a^2+bc}\geq\frac{(a+b+c)^2}{2\left(a^2b^2+b^2c^2+c^2a^2\right)+abc\,(a+b+c)}$$

170. Let a, b, c be positive real numbers. Prove that:

$$\sqrt{(a^2+b^2+c^2)(ab^3+bc^3+ca^3)} \geq abc + \frac{2\left(ab^3+bc^3+ca^3\right)}{a+b+c}$$

171. Let a, b, c be positive real numbers. Prove that:

$$1+\frac{ab^2+bc^2+ca^2}{(ab+bc+ca)(a+b+c)} \geq \frac{4.\sqrt[3]{(a^2+ab+bc)(b^2+bc+ca)(c^2+ca+ab)}}{(a+b+c)^2}$$

172. Let a, b, c be positive real numbers. Prove that:

$$\frac{(a+b+c)^3}{8abc} \geq \frac{\left(a^2+ab+bc\right)\left(b^2+bc+ca\right)\left(c^2+ca+ab\right)}{(a^2+bc)(b^2+ca)(c^2+ab)}$$

173. Let a, b, c be real numbers in $[1, 3]$ satisfying $a+b+c=6$. Prove that:

$$24 \leq a^3+b^3+c^3 \leq 36.$$

174. Let a, b, c, d be nonnegative numbers satisfying $a+b+c+d=4$. Prove that:

$$\frac{ab}{c+d+4}+\frac{bc}{d+a+4}+\frac{cd}{a+b+4}+\frac{da}{b+c+4}+\frac{\sqrt{abcd}}{3} \leq 1$$

175. Given a, b, c, d be the real nonnegative numbers satisfying $a+b+c+d=3$. Prove that:

$$\frac{ab}{3b+c+d+3}+\frac{bc}{3c+d+a+3}+\frac{cd}{3d+a+b+3}+\frac{da}{3a+b+c+3} \leq \frac{1}{3}$$

176. Let a, b, c, d be the nonnegative numbers satisfying $a+b+c+d=4$. Prove that:

$$\sqrt{\frac{a}{a^2+3b^2+5c^2+7d^2}}+\sqrt{\frac{b}{b^2+3c^2+5d^2+7a^2}} + \sqrt{\frac{c}{c^2+3d^2+5a^2+7b^2}}+\sqrt{\frac{d}{d^2+3a^2+5b^2+7c^2}} \leq 1.$$

177. Let a, b, c be nonnegative numbers. Prove that:

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}+\frac{28(ab+bc+ca)}{(a+b+c)^3} \geq \frac{11}{a+b+c}$$

178. Let a, b, c be real numbes satisfying $a+b+c=3$. Prove that:

$$\frac{(a+b)^2}{4a+b^2+c^2}+\frac{(b+c)^2}{4b+c^2+a^2}+\frac{(c+a)^2}{4c+a^2+b^2} \leq 2$$

179. Given A, B, C are three sides of a triangle. Find Minimum:

$$P = 2\left(\sin A + \sin B + \sin C\right) + \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$$

180. Let a, b, c be positive real numbers satisfying $a + b + c = 3$. Prove that:

$$ab + bc + ca + \frac{1}{abc} \geq 3 + abc$$

181. Let a, b, c be real numbers satisfying $a^2 + b^2 + c^2 = 2\left(ab + bc + ca\right)$. Prove that:

$$\frac{|a-b|}{\sqrt{2ab+c^2}} + \frac{|b-c|}{\sqrt{2bc+a^2}} + \frac{|c-a|}{\sqrt{2ca+b^2}} \geq 2$$

182. Let a, b, c be positive real numbers satisfying $a^2b + b^2c + c^2a = 3$ and $k \geq 7$. Prove that:

$$\frac{1}{a^3+k} + \frac{1}{b^3+k} + \frac{1}{c^3+k} \leq \frac{3}{k+1}$$

183. Given a, b, c are three real numbers satisfying $a + b + c = 3$; Prove that:

$$\frac{a^2+b^2c^2}{(b-c)^2} + \frac{b^2+c^2a^2}{(c-a)^2} + \frac{c^2+a^2b^2}{(a-b)^2} \geq 5$$

184. Given $a, b, c, k > 0$ and $a + b + c = 3$. Prove that:

$$\frac{a}{\sqrt{kb+c^2}} + \frac{b}{\sqrt{kc+a^2}} + \frac{c}{\sqrt{ka+b^2}} \geq \frac{3}{\sqrt{k+1}}$$

185. Given a, b, c are three reals; Prove that:

$$\left(\frac{a}{b-c}\right)^2 + \left(\frac{b}{c-a}\right)^2 + \left(\frac{c}{a-b}\right)^2 \geq \frac{1}{2} + \frac{3\left(ab+bc+ca\right)}{a^2+b^2+c^2}$$

186. Given a, b, c are three reals such that $a^2 + b^2 + c^2 = 3$; Prove that:

$$2(b^2c^2-a^2)(c^2a^2-b^2)(a^2b^2-c^2) + 64a^2b^2c^2 \geq (bc-a)^2(ca-b)^2(ab-c)^2$$

187. Given $a, b, c > 0$ and $a + b + c = 3$. Prove that

- $11\left(\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c}\right) + 3 \geq 12(ab+bc+ca)$
- $5\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \geq 4(ab+bc+ca) + 3$
- $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} \geq ab + bc + ca.$

188. Given $a, b, c > 0$. Prove that

$$\frac{a+b+c}{\sqrt[3]{abc}} + \frac{2(a+b+c)^4}{27(a^2b^2+b^2c^2+c^2a^2)} \geq 5$$

189. Let $a, b, c > 0$. Prove that:

$$\frac{2}{3\sqrt{3}} \cdot \sqrt{\frac{(a+b+c)^3}{abc}} \geq \frac{a(b+c)}{a^2+2bc} + \frac{b(c+a)}{b^2+2ca} + \frac{c(a+b)}{c^2+2ab}$$

190. Let a, b, c be positive real numbers. Prove that:

$$\frac{a+b}{a^2+bc+c^2} + \frac{b+c}{b^2+ca+a^2} + \frac{c+a}{c^2+ab+b^2} \geq \frac{27(ab^2+bc^2+ca^2+3abc)}{(a+b+c)^4}$$

191. Let a, b, c be positive real numbers. Prove that:

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}} \geq \frac{4}{3}\left(\frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab}\right)$$

192. Given $a, b, c > 0$ and $ab + bc + ca = 1$. Prove that:

$$(a^2+2bc)(b^2+2ac)(c^2+2ab) \geq \sqrt{1+36a^2b^2c^2[(a-b)(b-c)(c-a)]^2}$$

193. Let a, b, c be nonnegative numbers. Prove that:

$$\frac{a(bc+ca-2ab)}{(2a+b)^2} + \frac{b(ca+ab-2bc)}{(2b+c)^2} + \frac{c(ab+bc-2ca)}{(2c+a)^2} \geq 0$$

Equality occurs when $a = b = c$ or $(a, b, c) = (0, 1, 2)$ and its cyclics.

194. Let a, b, c be three real numbers and $k \in R$. Prove that:

$$\left(\frac{a}{4a-3b-c}\right)^2 + \left(\frac{b}{4b-3c-a}\right)^2 + \left(\frac{c}{4c-3a-b}\right)^2 \geq \frac{10}{169}$$

The general problem:

$$\left(\frac{a}{2ka-(k+1)b-(k-1)c}\right)^2 + \left(\frac{b}{2kb-(k+1)c-(k-1)a}\right)^2 + \left(\frac{c}{2kc-(k+1)a-(k-1)b}\right)^2 \geq \frac{2(1+k^2)}{(3k^2+1)^2}$$

195. Let a, b, c be nonnegative numbers. Prove that:

$$\frac{3}{2} + \frac{abc(a^{2008}+b^{2008}+c^{2008})}{(a+b+c)(a^{2010}+b^{2010}+c^{2010})} \geq 3 \cdot \sqrt{\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3}}$$

196. Let a, b, c be three real numbers saitsfy $a + b + c = 3$. Prove that:

$$\frac{1}{3a^2+4b^2+5c^2+6} + \frac{1}{3b^2+4c^2+5a^2+6} + \frac{1}{3c^2+4a^2+5b^2+6} \leq \frac{1}{6}$$

197. Let a, b, c be three real numbers. Prove that:

$$\frac{a+b}{2a^2+3b^2+3c^2+8}+\frac{b+c}{2b^2+3c^2+3a^2+8}+\frac{c+a}{2c^2+3a^2+3b^2+8}\le\frac{3}{8}$$

198. Let a, b, c be positive numbers. Prove that:

$$\frac{\sqrt{a(a^2+ab+bc)}}{a+b}+\frac{\sqrt{b(b^2+bc+ca)}}{b+c}+\frac{\sqrt{c(c^2+ca+ab)}}{c+a}\le\frac{3}{2}\sqrt{a+b+c}$$

199. Let a, b, c be nonnegative numbers. Prove that:

$$\frac{a(b^2+ca)}{(2a+b)^2}+\frac{b(c^2+ab)}{(2b+c)^2}+\frac{c(a^2+bc)}{(2c+a)^2}\ge\frac{3}{160}\cdot\frac{(2a+5b)(2b+5c)(2c+5a)-28abc}{(a+b+c)^2}$$

200. Let a, b, c be real numbers satisfying $a+b+c=3$. Prove that:

$$\frac{|a+b|}{4a+b^2+c^2}+\frac{|b+c|}{4b+c^2+a^2}+\frac{|c+a|}{4c+a^2+b^2}\le 1$$

2 Solutions To Selected Problems

1. **For positive reals a, b, c prove that:**

$$(a+b+c)^3 \geq 6\sqrt{3}(a-b)(b-c)(c-a)$$

Solution
Without any loss of generality, assume $c \geq b \geq a$. Then we have

$$(a-b)(b-c)(c-a) = (c-a)(b-a)(c-b) \leq bc(c-b);$$

And it suffices to prove that

$$(b+c)^3 \geq 6\sqrt{3}bc(c-b).$$

Now, using a simple balancing method we can find out that AM-GM in the following manner proves it easily,

$$2bc(c-b) = \left(\sqrt{3}+1\right) b \cdot \left(\sqrt{3}-1\right) c \cdot (c-b) \leq \frac{\left\{\sqrt{3}(b+c)\right\}^3}{27} = \frac{(b+c)^3}{3\sqrt{3}}.$$

Equality occurs if and only if $(a, b, c) = \left(0,\ k(\sqrt{3}-1),\ k(\sqrt{3}-1)\right)$ and its cyclic permutations. $\square$

2. **For $a, b, c \geq 0$ and $k \in \mathbf{R}$ find the best constant that satisfies**

$$(a+b+c)^5 \geq k(a^2+b^2+c^2)(a-b)(b-c)(c-a)$$

Solution
Letting $c = \sqrt[4]{5}+1, b = \sqrt[4]{5}-1, c = 0$; we obtain $k = 10\sqrt[4]{5}$. So it is sufficient to check the given inequality for $k = 10\sqrt[4]{5}$.
Assume WLOG that $c \geq b \geq a$.
Note that;

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} = 3 - \frac{(a-b)^2+(b-c)^2+(c-a)^2}{a^2+b^2+c^2} \geq 3 - \frac{b^2+c^2+(b-c)^2}{b^2+c^2} = \frac{(b+c)^2}{b^2+c^2};$$

So that it is enough to show that

$$(b+c)^5 \geq 10\sqrt[4]{5}(c-b)bc(b^2+c^2).$$

Note that we have, $bc(b^2+c^2) = \frac{1}{8}\left((c+b)^4 - (c-b)^4\right)$; so that we may rephrase the last inequality into

$$(b+c)^5 \geq \frac{10\sqrt[4]{5}}{8} \cdot (c-b)\left((c+b)^4 - (c-b)^4\right).$$

Due to homogeneity of this inequality, assume $c+b = 2\sqrt[4]{5}$, and $c-b = 2t$, where $t \geq 0$. Then our last inequality simplifies as

$$4 \geq t(5-t^4).$$

Note that using the AM-GM inequality, we have

$$t(5-t^4) \leq t(8-4t) = 4t(2-t) \leq (t+2-t)^2 = 4.$$

Hence we are done. Equality holds iff $(a,b,c) \sim \left(\left(\sqrt[4]{5}+1\right), \left(\sqrt[4]{5}-1\right), 0\right)$ and its relevant permutations. $\square$

3. **For nonnegative reals a, b, c, find the best k satisfying**

$$(a+b+c)^5 \geq k(ab+bc+ac)(a-b)(b-c)(c-a)$$

Solution

Again, WLOG assume that $c \geq b \geq a$. Putting $a = 0, b = \dfrac{\sqrt{5}-1}{2}, c = \dfrac{\sqrt{5}+1}{2}$; we obtain $k = 25\sqrt{5}$. So we have to prove that

$$(a+b+c)^5 \geq 25\sqrt{5}(ab+bc+ca)(b-a)(c-b)(c-a).$$

Note that

$$\begin{aligned}(ab+bc+ca)(bc-ca)(bc-ab) &\leq \frac{1}{27}(ab+bc+ca+bc-ca+bc-ab)^3 \\ &= b^3c^3;\end{aligned}$$

So that we get

$$(ab+bc+ca)(c-a)(b-a) \leq b^2c^2.$$

Hence it is sufficient to check that

$$(a+b+c)^5 \geq 25\sqrt{5}b^2c^2(c-b).$$

Note that, from the AM-GM inequality, we have

$$\begin{aligned}b^2c^2(c-b) &= \left(\left(\frac{\sqrt{5}+1}{2}\right)b\right)^2\left(\left(\frac{\sqrt{5}-1}{2}\right)c\right)^2(c-b) \\ &\leq \left(\frac{b(\sqrt{5}+1)+c(\sqrt{5}-1)+(c-b)}{5}\right)^5 \\ &= \frac{(b+c)^5}{25\sqrt{5}} \leq \frac{(a+b+c)^5}{25\sqrt{5}}.\end{aligned}$$

Equality holds if and only if $a = 0, b = \frac{\sqrt{5}-1}{2}k, c = \frac{\sqrt{5}+1}{2}k$, (where k is a positive constant) and its relevant permutations. $\square$

4. For nonnegative a, b, c, find the best k such that

$$(a^2 + b^2 + c^2)^3 \geq \; k(a-b)^2(b-c)^2(c-a)^2$$

Solution

Let us assume without loss of generality that $c \geq b \geq a$. Letting $a = 0, b = \sqrt{5} - 1, c = \sqrt{5} + 1$; we get $k = 27$. Hence it is sufficient to check the inequality for $k = 27$.

However, note that we have $(a-b)(b-c)(c-a) \leq bc(c-b)$; and it is enough to check that

$$a^2 + b^2 + c^2 \geq 3\sqrt[3]{(b^2 - 2bc + c^2)(bc)(bc)};$$

Which is obvious from the AM-GM inequality. Equality holds in the original inequality iff $(a, b, c) \sim \left(0, \sqrt{5} - 1, \sqrt{5} + 1\right)$ and all its *symmetric* permutations. $\square$

5. **For nonnegative reals a, b, c prove that**

$$\frac{ab}{(a+b)^2} + \frac{bc}{(b+c)^2} + \frac{ca}{(c+a)^2} \leq \frac{1}{4} + \frac{4abc}{(a+b)(b+c)(c+a)}$$

Solution

We can rewrite this inequality into;

$$\frac{(b-c)^2}{(b+c)^2} + \frac{(c-a)^2}{(c+a)^2} + \frac{(a-b)^2}{(a+b)^2} \geq \frac{2a(b-c)^2 + 2b(c-a)^2 + 2c(a-b)^2}{(a+b)(b+c)(c+a)};$$

Or,

$$\frac{(b-c)^2(a-b)(a-c)}{(b+c)^2(a+b)(c+a)} + \frac{(c-a)^2(b-c)(b-a)}{(c+a)^2(a+b)(b+c)} + \frac{(a-b)^2(c-a)(c-b)}{(a+b)^2(b+c)(c+a)} \geq 0;$$

Or,

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(b+c)^2(c+a)^2} \geq 0;$$

Which is perfectly true. Equality holds iff $a = b = c$. $\square$

6. **For** $a, b, c > 0$ **satisfying** $a^2 + b^2 + c^2 = 6$ **Find** P_{min} **where**

$$P = \frac{a}{bc} + \frac{2b}{ca} + \frac{5c}{ab}$$

Solution

Note that $\cdot P = \sqrt{\frac{a^2+b^2+c^2}{6}}\left(\frac{a}{bc} + \frac{2b}{ca} + \frac{5c}{ab}\right)$. Now, let $x = \frac{bc}{a}, y = \frac{ca}{b}, z = \frac{ab}{c}$ so that $xy + yz + zx = a^2 + b^2 + c^2$, and due to homogeneity we can assume $xy + yz + zx = 1$. Then it is enough to find the minimum value of the expression $P = \frac{1}{\sqrt{6}}\left(\frac{1}{x} + \frac{2}{y} + \frac{5}{z}\right)$.

Now, note that replacing x, y, z with $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we have to minimize, for $xyz = x + y + z$,

$$\sqrt{6}P = x + 2y + \frac{5(x+y)}{xy-1}.$$

However, note that

$$\begin{aligned}
\sqrt{6}P &= x + \frac{2}{x} + \frac{5}{x} + \frac{2(xy-1)}{x} + \frac{5(x+y)}{xy-1} - \frac{5}{x} \\
&= x + \frac{7}{x} + \frac{2(xy-1)}{x} + \frac{5(x^2+1)}{x(xy-1)} \\
&\geq x + \frac{7}{x} + 2\sqrt{\frac{10(xy-1)(x^2+1)}{x^2(xy-1)}} \\
&= x + \frac{7}{x} + 2\sqrt{10}\sqrt{1 + \frac{1}{x^2}} \\
&\geq x + \frac{7}{x} + 2\left(3 + \frac{1}{x}\right) \\
&= x + \frac{9}{x} + 6 \geq 12;
\end{aligned}$$

Where the last step follows from $x + \frac{9}{x} \geq 2\sqrt{x \cdot \frac{9}{x}} = 6$. Hence $P \geq 2\sqrt{6}$. Equality holds if and only if $a = \sqrt{2}, b = \sqrt{3}, c = 1$. $\square$

7. **For nonnegative reals** a, b, c **Prove that:**

$$\frac{(a+b)^2(a+c)^2}{(b^2-c^2)^2} + \frac{(b+c)^2(a+b)^2}{(c^2-a^2)^2} + \frac{(b+c)^2(c+a)^2}{(a^2-b^2)^2} \geq 2$$

Solution 1

Let $x = \frac{(a+b)(c+a)}{(b+c)(b-c)}, y = \frac{(b+c)(a+b)}{(c+a)(c-a)}, z = \frac{(b+c)(c+a)}{(a+b)(a-b)}$. Then we observe that

$$\begin{aligned} xy+yz+zx=\sum_{cyc}\frac{(a+b)^2(b+c)(c+a)}{(b+c)(c+a)(b-c)(c-a)} &= \sum_{cyc}\frac{(a+b)^2}{(b-c)(c-a)} \\ &= -\frac{(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)} \\ &= -1. \end{aligned}$$

And therefore,

$$(x+y+z)^2 \geq 0 \implies x^2+y^2+z^2 \geq -2(xy+yz+zx) = 2.$$

We are done. $\square$

Solution 2
Note that

$$\sum_{cyc}\frac{(a+b)^2(a+c)^2}{(b^2-c^2)^2} \geq \sum_{cyc}\frac{a^4}{(b^2-c^2)^2};$$

But

$$\sum_{cyc}\frac{a^4}{(b^2-c^2)^2} - 2 = \left(\sum \frac{a^2}{b^2-c^2}\right)^2 \geq 0.$$

$\square$

8. **For positive reals a, b, c prove that**

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{16}{5}\cdot\frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{18}{5}$$

Solution
Due to homogeneity assume that $a+b+c=1$. Letting $q = ab+bc+ca, r = abc$, the original inequality can be found to be equivalent to with

$$\frac{1-2q+3r}{q-r}+\frac{16q}{5(1-2q)} \geq \frac{18}{5}.$$

Note that $r \geq 0$ can help us in rephrasing this as $(4q-1)(18q-5) \geq 0$. When $q \leq \frac{1}{4}$; this holds good. Hence it is sufficient to check the case of $q \geq \frac{1}{4}$.
In this case, however, using the Schur's inequality of third degree, we have $r \geq \frac{4q-1}{9}$, so that it is enough to check that

$$\frac{1-2q+3\cdot\frac{4q-1}{9}}{q-\frac{4q-1}{9}}+\frac{16q}{5(1-2q)} \geq \frac{18}{5};$$

Which factorises into $(20q-3)(4q-1) \geq 0$, which is perfectly true due to our assumption $q \geq \frac{1}{4}$.
Equality holds in the original inequality if and only if $a = b = c$ or $a = b, c = 0$. $\square$

9. **For positive a, b, c; show that :**

$$\frac{(a^2+bc)(b^2+ca)(c^2+ab)}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} + \frac{(a-b)(a-c)}{b^2+c^2} + \frac{(b-c)(b-a)}{c^2+a^2} + \frac{(c-a)(c-b)}{a^2+b^2} \geq 1$$

Solution
Note that

$$\begin{aligned}
&(a^2+bc)(b^2+ca)(c^2+ab) - (a^2+b^2)(b^2+c^2)(c^2+a^2) \\
&= \sum_{cyc} \left(b^3c^3 + abc \cdot a^3 - a^4(b^2+c^2)\right) \\
&= \frac{1}{2} \sum_{cyc} \left(2b^3c^3 + 2abc \cdot a^3 - a^4(b^2+c^2) - b^2c^2(b^2+c^2)\right) \\
&= -\frac{1}{2}(a^4+b^2c^2)(b-c)^2;
\end{aligned}$$

So that it is sufficient to check that

$$\begin{aligned}
\sum_{cyc}(a^2+b^2)(c^2+a^2)(a-b)(a-c) &\geq \frac{1}{2} \sum_{cyc}(a^4+b^2c^2)(b-c)^2 \\
&= \sum_{cyc} \left(b^4+c^4+a^2(b^2+c^2)\right)(a-b)(a-c);
\end{aligned}$$

Which can be rephrased as

$$\sum_{cyc} \left(a^4+b^2c^2-b^4-c^4\right)(a-b)(a-c) \geq 0;$$

Or,

$$\sum_{cyc}(a+b)(a+c)(a-c)^2(a-b)^2 \geq 0.$$

Equality holds if and only if $a = b = c$. $\square$

10. **For positive reals a, b, c prove that:**

$$1 + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq \frac{16abc}{(a+b)(b+c)(c+a)}$$

Solution
Rewrite the given problem into,

$$\frac{ab+bc+ca}{a^2+b^2+c^2}-1 \geq 2\left(\frac{8abc}{(a+b)(b+c)(c+a)}-1\right);$$

Which can be rephrased as

$$-\frac{1}{2}\sum_{cyc}\frac{(b-c)^2}{a^2+b^2+c^2} \geq -2\sum_{cyc}\frac{a(b-c)^2}{(a+b)(b+c)(c+a)};$$

Which can be rephrased as

$$\sum_{cyc}(b-c)^2\left(\frac{4a}{(a+b)(b+c)(c+a)}-\frac{1}{a^2+b^2+c^2}\right) \geq 0.$$

Note that this is obvious from SOS. Equality holds if and only if $a=b=c$.
□

11. For nonnegative a, b, c prove that :

$$(a^2+b^2+c^2-1)^2 \geq 2(a^3b+b^3c+c^3a-1)$$

Solution
Using the famous inequality of Vasc, it is sufficient to check that

$$\left(a^2+b^2+c^2-1\right)^2 \geq \frac{2}{3}(a^2+b^2+c^2)^2-2;$$

Which, after letting $x=a^2+b^2+c^2$, reduces to

$$3(x^2-2x+1) \geq 2x^2-6; \iff (x-3)^2 \geq 0.$$

Equality holds if and only if $a=b=c=1$. □

12. **Let $a, b, c \geq 0$ satisfy $a+b+c=2$. Prove that we have;**

- $\left(\sqrt{a^3}+\sqrt{b^3}\right)\left(\sqrt{b^3}+\sqrt{c^3}\right)\left(\sqrt{c^3}+\sqrt{a^3}\right) \leq 2;$
- $\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right) \leq 2;$
- $\left(\sqrt{a^5}+\sqrt{b^5}\right)\left(\sqrt{b^5}+\sqrt{c^5}\right)\left(\sqrt{c^5}+\sqrt{a^5}\right) \leq 2;$
- $\left(a^3+b^3\right)\left(b^3+c^3\right)\left(c^3+a^3\right) \leq 2;$
- $\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right) \leq (a+b)(b+c)(c+a).$

Solution

PART (a) : $\boxed{\left(\sqrt{a^3}+\sqrt{b^3}\right)\left(\sqrt{b^3}+\sqrt{c^3}\right)\left(\sqrt{c^3}+\sqrt{a^3}\right)\le 2}$

After using the Cauchy-Schwarz inequality, we have to show that

$$(a+b)(b+c)(c+a)(a^2+b^2)(b^2+c^2)(c^2+a^2)\le 4.$$

Without loss of generality, assume that $c=\min\{a,\, b,\, c\}$. With this assumption, we have

$$b^2+c^2\le b^2+bc=b(b+c),\quad c^2+a^2\le ca+a^2=a(a+c).$$

Therefore, it suffices to prove that

$$ab(a^2+b^2)(a+b)(a+c)^2(b+c)^2\le 4.$$

Now, by the AM-GM inequality, we obtain

$$ab(a^2+b^2)=\frac{1}{2}\cdot 2ab\cdot(a^2+b^2)\le\frac{1}{2}\cdot\left[\frac{2ab+(a^2+b^2)}{2}\right]^2=\frac{(a+b)^4}{8}.$$

Thus, to complete the proof, we must prove that

$$(a+b)^5(a+c)^2(b+c)^2\le 32.$$

However, this is true according to the AM-GM inequality

$$\begin{aligned}(a+b)^5(a+c)^2(b+c)^2&=\frac{1}{16}\cdot(a+b)^5\cdot[2(a+c)]^2\cdot[2(b+c)]^2\\&\le\frac{1}{16}\left[\frac{5\cdot(a+b)+2\cdot 2(a+c)+2\cdot 2(b+c)}{9}\right]^9\\&=\frac{1}{16}\left(\frac{9a+9b+8c}{9}\right)^9\le\frac{1}{16}\left(\frac{9a+9b+9c}{9}\right)^9=32.\end{aligned}$$

We are done. □

PART (b) : $\boxed{\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)\le 2}$

Without loss of generality, assume $c=\min\{a,b,c\}$. Then, we note that

$$(b^2+c^2)(c^2+a^2)\le b(b+c)\cdot a(c+a)=ab(b+c)(c+a).$$

So, it is sufficient to check that

$$ab(b+c)(c+a)(a^2+b^2)\le 2.$$

Note that

$$\begin{aligned}
4ab(b+c)(c+a)(a^2+b^2) &\le \left[\frac{2a(b+c)+2b(c+a)+a^2+b^2}{3}\right]^3 \\
&\le \left[\frac{2bc+(a+b+c)^2}{3}\right]^3 \\
&\le \left[\frac{\frac{1}{2}(b+c)^2+(a+b+c)^2}{3}\right]^3 \\
&\le 8.
\end{aligned}$$

So, we are done. □

Part (c) : $\boxed{\left(\sqrt{a^5}+\sqrt{b^5}\right)\left(\sqrt{b^5}+\sqrt{c^5}\right)\left(\sqrt{c^5}+\sqrt{a^5}\right) \le 2}$

Note that

$$\left(\sqrt{a^5}+\sqrt{b^5}\right)^2 \le \left(a^2+b^2\right)\left(a^3+b^3\right);$$

So we will be done if part (d) also holds good. □

Part (d) : $\boxed{\left(a^3+b^3\right)\left(b^3+c^3\right)\left(c^3+a^3\right) \le 2}$

Again let us assume that $c = \min\{a,b,c\}$. Note that we have

$$\left(a^3+b^3\right)\left(b^3+c^3\right)\left(c^3+a^3\right) = (a+b)(b+c)(c+a)\prod_{cyc}(a^2-ab+b^2).$$

From our assumption, we have $(b^2-bc+c^2)(c^2-ca+a^2) \le a^2b^2$. Thus, it suffices to check that

$$a^2b^2(a+b)(b+c)(c+a)(a^2-ab+b^2) \le 2$$

But, from the AM-GM inequality, we obtain

$$\begin{aligned}
a^2b^2(a+b)(b+c)(c+a)(a^2-ab+b^2) &= (a+b)\cdot(ab)^2(c^2+ab+bc+ac)(a^2-ab+b^2) \\
&\le (a+b)\left[\frac{2ab+c^2+ab+bc+ac+a^2-ab+b^2}{4}\right]^4 \\
&= (a+b)\left[\frac{a^2+b^2+c^2+2ab+bc+ca}{4}\right]^4 \\
&\le (a+b+c)\cdot\left[\frac{(a+b+c)^2}{4}\right]^4 \\
&= 2.
\end{aligned}$$

Hence we are done. □

Part (e) : $\boxed{\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right) \leq (a+b)(b+c)(c+a)\,.}$

It is obvious enough that we may to assume $c = \min\{a, b, c\}$. So, we get $(b^2+c^2)(c^2+a^2) \leq ab(b+c)(c+a)$. Hence it is enough to check that

$$ab(a^2+b^2) \leq a+b;$$

Which is perfectly true, since, according to the AM-GM inequality we obtain

$$ab(a^2+b^2) = \frac{1}{8}\cdot 4(2ab)(a^2+b^2) \leq \frac{1}{8}(a^2+b^2+2ab)^2 \leq (a+b)\left(\frac{a+b+c}{2}\right)^3$$
$$= a+b.$$

We are done. □

In all the inequalities from (a) to (e), equality occurs if and only if $(a, b, c) = (1, 1, 0)$ and its cyclic permutations. □

13. $a, b, c \geq 0$**. Prove that**

$$(a^2+5bc)(b^2+5ca)(c^2+5ab) \geq 27abc(a+b)(b+c)(c+a)$$

Solution
After expanding, we see that it is enough to check

$$5\sum_{cyc} a^3b^3 + 25\sum_{cyc} a^4bc + 72a^2b^2c^2 \geq 27abc\sum_{cyc} a^2(b+c)$$

Using Schur's inequality of third degree,

$$5\sum_{cyc}\left(a^3b^3+3a^2b^2c^2\right) \geq 5abc\sum_{cyc} a^2(b+c);$$

Hence it suffices to prove that;

$$25\sum a^4bc + 57abc \geq 22abc\sum a^2(b+c).$$

But this is also obvious due to Schur of the third degree and AM-GM:

$$25\sum a^4bc + 57abc \geq 22abc(\sum a^3 + 3abc) \geq 22abc\sum a^2(b+c).$$

Hence we are done. □

14. **Let** $a, b, c \geq 0$**. Prove that:**

$$(2a^2 + 7bc)(2b^2 + 7ca)(2c^2 + 7ab) \geq 27(ab + bc + ca)^3$$

Solution
After expanding, we have to show that

$$\sum_{cyc} a^3b^3 + 98\sum_{cyc} a^4bc + 189a^2b^2c^2 \geq 81abc\sum_{cyc} a^2(b + c).$$

We may rewrite the inequality into

$$\left(\sum_{cyc} a^3b^3 + 3a^2b^2 - abc\sum_{cyc} a^2(b + c)\right) + abc\left(98\sum_{cyc} a^3 + 186abc - 80\sum_{cyc} a^2(b + c)\right) \geq$$
$$\left(\sum_{cyc} a^3b^3 + 3a^2b^2 - abc\sum_{cyc} a^2(b + c)\right) + 80abc\left(\sum_{cyc} a^3 + 3abc - \sum_{cyc} a^2(b + c)\right);$$

Which is perfectly true using the AM-GM inequality. □

15. $a, b, c \geq 0$ **are the sides of a triangle. Prove that**

$$a^3 + b^3 + c^3 + 9abc \leq 2[ab(a + b) + bc(b + c) + ca(c + a)]$$

Solution
Rewrite this into

$$S = \sum_{cyc}(b + c - a)(a - b)(a - c) \geq 0.$$

Assume without loss of generality that $a \geq b \geq c$, so that we have $a + b - c \geq c + a - b \geq b + c - a \geq 0$. Again, using $a - c \geq a - b \geq 0$, we get

$$S \geq (b+c-a)(a-b)(a-c) + (b-c)[(a-c)(a+b-c) - (a-b)(a+c-b)] \geq 0.$$

□

16. **Let** $a, b, c > 0$**. Show that:**

$$\frac{a^2}{2a^2 + (b + c - a)^2} + \frac{b^2}{2b^2 + (c + a - b)^2} + \frac{c^2}{2c^2 + (a + b - c)^2} \leq 1$$

Solution
Let us rewrite the given inequality into

$$\frac{(b + c - a)^2}{2a^2 + (b + c - a)^2} + \frac{(c + a - b)^2}{2b^2 + (c + a - b)^2} + \frac{(a + b - c)^2}{2c^2 + (a + b - c)^2} \geq 1.$$

Note that using the Cauchy-Schwarz inequality, we have

$$\sum_{cyc}\left(\frac{(b+c-a)^4}{2a^2(b+c-a)^2+(b+c-a)^4}\right) \geq \frac{\sum(b+c-a)^2}{\sum(b+c-a)^4+2\sum a^2(b+c-a)^2};$$

So that it is enough to prove that:

$$\frac{\sum(b+c-a)^2}{\sum(b+c-a)^4+2\sum a^2(b+c-a)^2} \geq 1;$$

Which equivalents

$$4\sum_{cyc} a^4 + 4\sum_{cyc} a^2bc \geq 4\sum_{cyc} a^3(b+c)$$

Which is nothing other than the Schur's inequality of fourth degree. $\square$

17. **Let** $a, b, c \geq 0$ **Show that:**

$$\frac{3a^2+5ab}{(b+c)^2} + \frac{3b^2+5bc}{(c+a)^2} + \frac{3c^2+5ca}{(a+b)^2} \geq 6$$

Solution
Let us rewrite this inequality into

$$3\sum_{cyc}\frac{a(a+b)}{(b+c)^2} + 2\sum_{cyc}\frac{ab}{(b+c)^2} \geq 6.$$

Note that the sequences $\{bc, ca, ab\}$, and $\left\{\frac{1}{(b+c)^2}, \frac{1}{(c+a)^2}, \frac{1}{(a+b)^2}\right\}$ are oppositely sorted, hence we get

$$2\sum_{cyc}\frac{ab}{(b+c)^2} \geq \sum_{cyc}\frac{ab+bc}{(b+c)^2} = \sum_{cyc}\frac{a(b+c)}{(a+b)^2}.$$

Now, observe that

$$2\cdot\frac{a(a+b)}{(b+c)^2} + \frac{a(b+c)}{(a+b)^2} \overset{\text{AM-GM}}{\geq} 3\sqrt[3]{\frac{a^3(a+b)^2(b+c)}{(b+c)^4(a+b)^2}} = 3\cdot\frac{a}{b+c}.$$

Also,

$$\sum_{cyc}\frac{a^2}{(b+c)^2} + \sum_{cyc}\frac{ab}{(b+c)^2} \overset{\text{Rearrangement}}{\geq} \sum_{cyc}\frac{ca}{(b+c)^2} + \sum_{cyc}\frac{ab}{(b+c)^2}$$
$$= \sum_{cyc}\frac{a}{b+c};$$

So, adding these two, we have to show that

$$4\sum_{cyc}\frac{a}{b+c}\ge 6;$$

Which is obvious from Nessbitt's inequality. $\square$

18. **Let** $a,b,c>0$ **satisfy** $a+b+c=3$**. Prove that:**

$$(a^3+b^3+c^3)(ab+bc+ca)^8\le 3^9$$

Solution
Using the AM-GM inequality,we have

$$(a^3+b^3+c^3)(ab+bc+ca)^8\le\left[\frac{(a^3+b^3+c^3)+8(ab+bc+ca)}{9}\right]^9.$$

Again, using the well-know inequality $8(a+b+c)(ab+bc+ac)\le 9(a+b)(b+c)(c+a)$; we get

$$ab+bc+ca\le\frac{3}{8}(a+b)(b+c)(c+a);$$

And

$$\frac{(a^3+b^3+c^3)+8(ab+bc+ca)}{9}\le\left(\frac{(a^3+b^3+c^3)+3(a+b)(b+c)(c+a)}{9}\right)^9$$
$$=3^9.$$

Hence we are done. Equality holds iff $a=b=c=1$. $\square$

19. $a,b,c\ge 0$ **Show that**

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}+\frac{10abc}{(a+b)(b+c)(c+a)}\ge 2$$

Solution
Let $x=\frac{a}{b+c}, y=\frac{b}{c+a}, z=\frac{c}{a+b}$. Then we have to show that

$$x^2+y^2+z^2+10xyz\ge 2.$$

Firstly, we have the identity $xy+yz+zx+2xyz=1$. Secondly, using the Schur's inequality of third degree, we have

$$x^2+y^2+z^2+6xyz+4xyz\overset{x+y+z\ge\frac{3}{2}}{\ge}x^2+y^2+z^2+\frac{9xyz}{x+y+z}+4xyz$$
$$\ge 2(xy+yz+zx)+4xyz=2.$$

We are done. Equality holds iff $a = b = c$. $\square$

20. **For $a, b, c \geq 0$ such that $a+b+c = 3$, prove the following inequality:**

$$(ab^3 + bc^3 + ca^3)(ab + bc + ca) \leq 16$$

Solution
Note that using the AM-GM inequality, we have:

$$\frac{\left(ab^3 + bc^3 + ca^3\right)^2}{4} \cdot (ab + bc + ca)^2 \leq \frac{1}{27}\left[ab^3 + bc^3 + ca^3 + (ab + bc + ca)^2\right]^3;$$

So that it is sufficient to check that

$$ab^3 + bc^3 + ca^3 + (ab + bc + ca)^2 \leq \frac{4}{27}(a + b + c)^4.$$

Note that,

$$ab^3 + bc^3 + ca^3 + (ab + bc + ca)^2 = (a + b + c)(a^2b + b^2c + c^2a + abc);$$

So that we have to show

$$a^2b + b^2c + c^2a + abc \leq \frac{4}{27}(a + b + c)^3;$$

Which is a well-known inequality. Equality holds in the original inequality if and only if $(a, b, c) = (0, 1, 2)$ or its cyclics. $\square$

21. $a, b, c > 0$ **satisfy** $abc = 1$**. Prove that**

$$\frac{a}{\sqrt{b^2 + 2c}} + \frac{b}{\sqrt{c^2 + 2a}} + \frac{c}{\sqrt{a^2 + 2b}} \geq \sqrt{3}$$

Solution
Note that using the Hölder's inequality, we have

$$\left(\sum_{cyc} \frac{a}{\sqrt{b^2 + 2c}}\right)^2 \sum_{cyc} a(b^2 + 2c) \geq (a + b + c)^3;$$

So that it is sufficient to check that

$$(a + b + c)^3 \geq 3\sum_{cyc} a(b^2 + 2c);$$

Which can be rephrased as

$$a^3 + b^3 + c^3 + 3(a^2b + b^2c + c^2a) + 6abc \geq 6(ab + bc + ca);$$

Which is perfectly true due to a perpetual use of the AM-GM inequality as:

$$\begin{aligned}\sum_{cyc}(b^3 + a^2b) + 2\sum_{cyc} a^2b + 6 &\geq 2\sum_{cyc} ab^2 + 2\sum_{cyc} a^2b + 6 \\ &= 2\sum_{cyc}\left(ab^2 + a^2b + 1\right) \\ &\geq 6(ab + bc + ca).\end{aligned}$$

Equality holds iff $a = b = c = 1$. □

22. For nonnegative a, b, c satisfying $ab + bc + ca = 3$, prove that

$$3(a + b + c) + 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \geq 15$$

Solution
Note that applying the AM-GM inequality, we have

$$3(a + b + c) + 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \geq 5\sqrt[5]{(a + b + c)^3\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2}.$$

Hence it is enough to check that

$$(a + b + c)^3\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \geq 3^5 = 27(ab + bc + ca)^2.$$

Since the last inequality is homogeneous, we can, without loss of generality, dump the previous condition $ab+bc+ca = 3$ and assume that $a+b+c = 3$. Then, our last inequality rewrites as

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca;$$

Which is the well-known Russia 2002 inequality. In order to prove this we can note that $ab + bc + ca = \dfrac{9 - a^2 - b^2 - c^2}{2}$ and rewrite this as

$$\sum_{cyc}\left(a^2 + 2\sqrt{a}\right) \geq 3a + 3b + 3c;$$

Which is perfectly true due to the AM-GM inequality. Equality holds if and only if $a = b = c = 1$. □

23. **For nonnegative reals a, b, c prove that:**

$$\frac{a^2}{(b + c)^2} + \frac{b^2}{(c + a)^2} + \frac{c^2}{(a + b)^2} + \frac{1}{2} \geq \frac{5}{4} \cdot \frac{a^2 + b^2 + c^2}{ab + bc + ca}$$

Solution
Multiplying bot sides with $(a+b)(b+c)(c+a)$, we have to prove that

$$\sum_{cyc} \frac{a^2(a+b)(a+c)}{b+c} + \frac{(a+b)(b+c)(c+a)}{2} \geq \frac{5(a^2+b^2+c^2)(a+b)(b+c)(c+a)}{4(ab+bc+ca)}.$$

This maybe rewritten as

$$\sum_{cyc} \frac{a^2(a-b)(a-c)}{b+c} + abc\left[\frac{5(a^2+b^2+c^2)}{4(ab+bc+ca)}+1\right] + \frac{3}{4}\sum_{cyc} a^3 \geq \frac{3}{4}\sum_{cyc} ab(a+b)$$

Using the Vornicu-Schur inequality, we have $\sum_{cyc} \frac{a^2(a-b)(a-c)}{b+c} \geq 0$.
Also, from Schur's inequality of third degree we have
$\sum_{cyc} ab(a+b) \leq a^3+b^3+c^3+3abc$. So it suffices to check that

$$\frac{5}{4}\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}+1 \geq \frac{9}{4};$$

Which is obvious. Equality holds if and only if $a=b=c$ or $a=b, c=0$ and its cyclic permutations. $\square$

24. **For nonnegative a, b, c, show that**

$$3(a^4+b^4+c^4)+7(a^2b^2+b^2c^2+c^2a^2) \geq 2(a^3b+b^3c+c^3a)+8(ab^3+bc^3+ca^3)$$

Solution
In general, from the SOS technique of Can Vo Quoc Ba, we have the following inequality for all $a, b, c, m, n, p, g \in \mathbb{R}$ such that $\{m > 0\} \wedge \{3m(m+n) \geq p^2+pg+g^2\}$:

$$m\sum_{cyc} a^4 + n\sum_{cyc} a^2b^2 + p\sum_{cyc} a^3b + g\sum_{cyc} ab^3 - (m+n+p+g)\sum a^2bc \geq 0.$$

In this case, note that $m=3, n=7, p=-2, g=-8$ Which satisfy;

$$3m(m+n) = 90 > 4+16+64 = p^2+pg+g^2.$$

We are done. $\square$

25. **For $a, b, c \geq 0$, show that:**

$$\frac{a^4}{(a+b)^4} + \frac{b^4}{(b+c)^4} + \frac{c^4}{(c+a)^4} + \frac{3abc}{2(a+b)(b+c)(c+a)} \geq \frac{3}{8}$$

Solution
We will show that the stronger inequality holds,

$$\frac{a^4}{(a+b)^4}+\frac{b^4}{(b+c)^4}+\frac{c^4}{(c+a)^4}+\frac{2abc}{(a+b)(b+c)(c+a)}\geq\frac{7}{16}.$$

Let us substiute $x=\frac{b}{a}, y=\frac{c}{b}, z=\frac{a}{c}$ so that $xyz=1$; and we have to show that

$$\frac{1}{(1+x)^4}+\frac{1}{(1+y)^2}+\frac{1}{(1+z)^2}+\frac{2}{(1+x)(1+y)(1+z)}\geq\frac{7}{16}.$$

Without loss of generality, let us assume that $(1-x)(1-y)\geq 0 \implies 1+xy\geq x+y$. Therefore we get

$$(1+x)(1+y)\leq 2(1+xy)=2\left(1+\frac{1}{z}\right)=\frac{2(z+1)}{z}.$$

Also, we have the following inequality:

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}-\frac{1}{1+xy}=\frac{xy(x-y)^2+(1-xy)^2}{(1+x)^2(1+y)^2(1+xy)}\geq 0.$$

Using this in accordance with the Cauchy-Schwarz inequality, we obtain

$$\frac{1}{(1+x)^4}+\frac{1}{(1+y)^4}\geq\frac{1}{2}\left(\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}\right)^2\geq\frac{1}{2}\left(\frac{1}{1+xy}\right)^2=\frac{z^2}{2(z+1)^2}.$$

So, our last inequality may be rewritten into

$$\frac{z^2}{2(z+1)^2}+\frac{1}{(z+1)^4}+\frac{z}{(z+1)^2}\geq\frac{7}{16};$$

Which is perfectly true, being equivalent to with

$$\frac{(z+3)^2(z-1)^2}{2(z+1)^4}\geq 0.$$

Equality holds if and only if $a=b=c$. □

26. **Let** $a,b,c\geq 0$ **satisfy** $a+b+c=3$ **Prove that:**

$$\sqrt[3]{\frac{a^3+4}{a^2+4}}+\sqrt[3]{\frac{b^3+4}{b^2+4}}+\sqrt[3]{\frac{c^3+4}{c^2+4}}\geq\ 3$$

Solution 1
Note that using the Hölder's inequality, we have

$$\left(\sum_{cyc}\sqrt[3]{\frac{a^3+4}{a^2+4}}\right)^3\sum_{cyc}\frac{a^2+4}{a^3+4}\geq 27;$$

So that it is enough to check that

$$\frac{a^2+4}{a^3+4}+\frac{b^2+4}{b^3+4}+\frac{c^2+4}{c^3+4}\leq 3.$$

But, note that

$$\frac{a^2+4}{a^3+4}-1-\frac{1}{5}(1-a)=\frac{(a-1)^2(a^2-4a-4)}{(a^2+4)(a^3+4)}\leq 0;$$

So we obtain our desired result due to $a+b+c=3$. $\square$

Solution 2
Note that using the AM-GM inequality, we get

$$\sqrt[3]{\frac{a^3+4}{a^2+4}}+\sqrt[3]{\frac{b^3+4}{b^2+4}}+\sqrt[3]{\frac{c^3+4}{c^2+4}}\geq\ 3\sqrt[9]{\frac{(a^3+4)(b^3+4)(c^3+4)}{(a^2+4)(b^2+4)(c^2+4)}};$$

So that it is enough to check that

$$(a^3+4)(b^3+4)(c^3+4)\geq(a^2+4)(b^2+4)(c^2+4).$$

Note that from Holder's inequality, we have $5(a^3+4)^2\geq(a^2+4)^3$. Hence it suffices to check that

$$(a^2+4)(b^2+4)(c^2+4)\geq 5^3.$$

Assume $c=\min\{a,b,c\}\leq 1$. Note that using the Cauchy-Schwarz inequality, we have

$$(a^2+1+3)(1+b^2+3)(c^2+4)\geq(a+b+3)^2(c^2+4)=(6-c)^2(c^2+4).$$

Hence it is sufficient to show that

$$(6-c)^2(c^2+4)\geq 125,\iff(c-1)^2(c^2-10c+19)\geq 0.$$

We are done, since $c^2-10c+19=(c-5-\sqrt{6})(c-5+\sqrt{6})\geq 0$. Equality holds iff $a=b=c=1$. $\square$

27. **For positive reals a,b,c show that:**

$$5+\frac{3abc}{a^3+b^3+c^3}\geq\ 4\left(\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\right)$$

Solution
WLOG due to symmetry, assume $a\geq b\geq c$. Then the given inequality may be rewrritten into

$$2\sum_{cyc}\frac{(a-b)^2}{a^2+b^2}\geq\frac{a+b+c}{a^3+b^3+c^3}\left[(a-c)^2+(b-a)(b-c)\right].$$

From the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(b-c)^2}{b^2+c^2}+\frac{(a-c)^2}{c^2+a^2}\geq\frac{2(a-c)^2}{a^2+b^2+c^2}.$$

So, our last inequality may be rewritten into

$$(a-c)^2\left(\frac{4}{a^2+b^2+c^2}-\frac{a+b+c}{a^3+b^3+c^2}\right)+(a-b)(b-c)\frac{a+b+c}{a^3+b^3+c^3}\geq 0;$$

Which is perfectly true. Equality holds if and only if $a=b=c$. $\square$

28. **For nonnegative reals** a,b,c **prove that:**

$$\frac{(a-b)^2}{(a+b)^2}+\frac{(b-c)^2}{(b+c)^2}+\frac{(c-a)^2}{(c+a)^2}+\frac{24\,(ab+bc+ca)}{(a+b+c)^2}\leq\ 8$$

Solution

We can rephrase this inequality into:

$$\sum_{cyc}\frac{(b-c)^2}{(b+c)^2}\leq 8\left[1-\frac{3(ab+bc+ca)}{(a+b+c)^2}\right];$$

Or,

$$\sum_{cyc}\frac{(b-c)^2}{(b+c)^2}\leq\frac{4}{(a+b+c)^2}\left[(b-c)^2+(c-a)^2+(a-b)^2\right];$$

Which can be rephrased as

$$\sum_{cyc}(b-c)^2\left[\frac{4}{(a+b+c)^2}-\frac{1}{(b+c)^2}\right]\geq 0;\iff\sum_{cyc}S_a(b-c)^2\geq 0.$$

Note that

$$a^2S_a+b^2S_b+c^2S_c=\frac{4(a^2+b^2+c^2)}{(a+b+c)^2}-\sum_{cyc}\frac{a^2}{(b+c)^2};$$

So by the SOS Method, it suffices to check that we have the following inequality:

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}\leq\frac{4(a^2+b^2+c^2)}{(a+b+c)^2}.$$

Now, we will consider two cases, assuming $a=\max\{a,b,c\}$.

Case 1. $\boxed{3a^2\leq 3b^2+3c^2+8bc}$

Note that we have $\dfrac{a^2}{(b+c)^2} \leq \dfrac{4a^2}{(a+b+c)^2}$ in this case, and the rest is obvious.

Case 2. $\boxed{3a^2 > 3b^2 + 3c^2 + 8bc}$

In this case, we note that $\dfrac{1}{(b+c)^2} = \max\left\{\dfrac{1}{(b+c)^2}, \dfrac{1}{(c+a)^2}, \dfrac{1}{(a+b)^2}\right\}$, and so we obtain

$$\sum_{cyc} \frac{(b-c)^2}{(b+c)^2} \leq \frac{1}{(b+c)^2} \sum_{cyc} (b-c)^2;$$

And so it suffices to check that

$$(a+b+c)^2 \leq 4(b+c)^2$$